THERMAL CONDUCTIVITY OF AN INFINITE HOLLOW CYLINDER

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Inzhenerno-Fizicheskii Zhurnal, Vol. 9, No. 6, pp. 775–777, 1965

UDC 536.12

The axisymmetric temperature field of an infinite hollow cylinder with mixed boundary conditions is examined.

On the surface $r = r_2$ let the Newtonian heat transfer be given only on the section $|z| < a$, while on the section $|z| > a$ the cylinder surface interacts with the medium, whose temperature is zero.

On the inner surface of the cylinder boundary conditions of the third kind are specified, and the temperature of the medium is likewise zero.

We write the boundary conditions in the form

$$\alpha_1 T - \lambda_1 \frac{\partial T}{\partial r} = 0, \quad |z| < \infty, \quad r = r_1; \tag{1}$$

$$\alpha_2 T + \lambda_2 \frac{\partial T}{\partial r} = 0, \quad |z| > a, \quad r = r_2; \tag{2}$$

$$\alpha_3 T + \lambda_2 \frac{\partial T}{\partial r} \lambda_2 \varphi(z), \quad |z| < a, \quad r = r_2. \tag{3}$$

We shall examine the case when α_1 = const. All the conclusions may be extended without appreciable change to the case when α_3 depends on z.

We shall determine the temperature distribution function T(r, z) satisfying the Laplace equation in cylindrical coordinates

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{4}$$

and boundary conditions (1), (2), and (3).

It is easy to verify that the function

$$T(r, z) = [AI_0(mr) + BK_0(mr)]\cos mz + [CI_0(mr) + DK_0(mr)]\sin mz \tag{5}$$

satisfies (4) for any A, B, C, D, and arbitrary parameter m.

Satisfying (1) and equating coefficients with cos mz and sin mz to zero, we find that

$$A = BN, \quad C = DN, \tag{6}$$

where

$$N = (\alpha_1 K_0(mr_1) + \lambda_1 mK_1(mr_1))/(\alpha_1 I_0(mr_1) + \lambda_1 mI_1(mr_1)).$$

On the basis of (6), function (5) takes the form

$$T(r, z) = [NI_0(mr) + K_0(mr)][B\cos mz + D\sin mz].$$

We seek a solution of the problem in the form of the integral

$$T(r, z) = \int_0^\infty T(r, z, m)\, dm.$$

Satisfying the boundary conditions (2) and (3), we form the pair of integral equations

$$\int_0^\infty M(m)\cos mzdm + \int_0^\infty E(m)\sin mzdm = 0, \quad |z| > a; \tag{7}$$

$$\int_0^\infty M(m)[1 + P]\cos mzdm + \int_0^\infty E(m)[1 + P]\sin mzdm = \lambda_3 \varphi(z), \quad |z| < a, \tag{8}$$

where

$$M(m) = B\{\alpha_2[NI_0(mr_2) + K_0(mr_2)] + \lambda_2[NmI_1(mr_2) + mK_1(mr_2)]\},$$

$$E(m) = D\{\alpha_2[NI_0(mr_2) + K_0(mr_2)] + \lambda_2[NmI_1(mr_2) + mK_1(mr_2)]\},$$

$$P = (\alpha_3 - \alpha_2)[NI_0(mr_2) + K_0(mr_2)] \times \{\alpha_2[NI_0(mr_2) + K_0(mr_2)] + \lambda_2 m[NI_1(mr_2) - K_1(mr_2)]\}^{-1}.$$

We seek a solution of the integral equations in the form

$$M(m) = \frac{1}{\pi}\int_{-a}^{+a} f(t)\cos mtdt, \tag{9}$$

$$E(m) = \frac{1}{\pi}\int_{-a}^{+a} f(t)\sin mtdt, \tag{10}$$

where $f(t)$ is an auxiliary function allowing representation in the form of Fourier integrals, i.e.,

$$\frac{1}{\pi}\int_0^\infty\left[\int_{-a}^{+a} f(t)\cos mt\cos mzdt + \int_{-a}^{+a} f(t)\sin mt\sin mzdt\right]dm = \begin{cases} 0 & |z| > a \\ f(z) & |z| < a \end{cases} \tag{11}$$

Substituting expressions (9) and (10) for M(m) and E(m) into (7) on the basis of (11), we can verify that (7) is satisfied identically.

If we substitute (9) and (10) into (8), we obtain

$$f(z) + \int_{-a}^{+a} f(t)\, G(z, t)\, dt = \lambda_2 \varphi(z),$$

where

$$G(z, t) = \frac{1}{\pi} \int_0^{\infty} P \cos [m(z - t)]\, dm.$$

Consequently, a Fredholm integral equation of the second kind is obtained for $f(z)$, and the temperature distribution in the cylinder is expressed in terms of $f(z)$ as

$$T(r, z) = \frac{1}{\pi} \int_0^{\infty} \{ \alpha_2 [N I_0 (m r_2) + K_0 (m r_2)] + \qquad (12)$$

$$+ \lambda_2 m [N I_1 (m r_2) - K_1 (m r_2)] \}^{-1} \times$$

$$\times [N I_0 (mr) + K_0 (mr)]\, dm \int_{-a}^{+a} f(t) \cos m(z - t)\, dt. \qquad \begin{matrix}(12)\\ \text{(cont'd)}\end{matrix}$$

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19 March 1965